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The topological cigar observables

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Abstract: We study the topologically twisted cigar, namely the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ superconformal field theory at arbitrary level, and find the BRST cohomology of the topologically twisted $N=2$ theory. We find a one to one correspondence between the spectrum of the twisted coset and singular vectors in the Wakimoto modules constructed over the $\mathrm{SL}(2,\mathbb{R})$ current algebra. The topological cigar cohomology is the crucial ingredient in calculating the closed string spectrum of topological strings on non-compact Gepner models.

Keywords: Topological Strings, BRST Symmetry, Conformal Field Models in String Theory.

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Contents

1. Introduction

String theory is able to describe dynamics on highly curved space-times. Using exactly solvable conformal field theories, Gepner constructed models with $N = 2$ supersymmetry in four dimensions, that describe string propagation on highly curved compact backgrounds which are special points in the moduli space of Calabi-Yau compactifications.

To code the non-trivial physics one is interested in, it often suffices to concentrate on a region near a singularity, which may be embedded in a non-compact Calabi-Yau manifold. It then becomes interesting to describe and study exact conformal field theories that describe special points in the "moduli space" of non-compact Calabi-Yau's.

Furthermore, if sufficient supersymmetry is preserved, the background allows us to study a BPS sector of the full string theory, the topological string theory, which can be defined for any flat $3+1$ dimensional background tensored with a central charge $c = 9$ and $N = 2$ superconformal field theory. The observables of the topological string theory lie in the cohomology of the BRST-operator of the twisted theory (one of the worldsheet supercharges of the untwisted model). We focus in this paper on the key ingredient to computing this cohomology, which is the calculation of the cohomology of the relevant non-rational conformal field theory.

A very large class of non-compact Gepner models (though not all [1, 2]) is obtained by tensoring $N = 2$ minimal models with $N = 2$ Liouville theories (either in heterotic or type II string theory — we concentrate on the latter here). Equivalently [3 − 6], we can tensor $N = 2$ minimal models with superconformal cigar conformal field theories. These theories come equipped with an $N = 2$ superconformal algebra on the worldsheet and one can construct four dimensional supersymmetric string backgrounds provided the internal CFT has a total central charge $c = 9$. This constraint leaves us with a very large class of non-compact Gepner models (see e.g. [7 − 13]). Note that in general, the level k of the cigar theory can take on integer, fractional, and even irrational values (when combined with yet another cigar theory at irrational central charge).

To be more concrete, we give a few simple examples that illustrate these general ideas. Lower dimensional or non-critical superstring backgrounds in even dimensions have been extensively studied since they were first discussed in [7]. The conformal field theory (CFT) that describes these non-singular non-critical string vacua are a product of the flat space theory in d dimensions and an $\mathcal{N} = 2$ supersymmetric generalization of the cigar background:

$$\mathbb{R}^{1,d-1} \times \left[\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\right]_k \times \dots \tag{1.1}$$

In the minimal case in which there are no other factors present, the level of the coset CFT is fixed to be a rational number by demanding absence of the conformal anomaly at the quantum level, i.e. by requiring $c = 15$. This fixes the supersymmetric level k in terms of the dimension d

$$\frac{3d}{2} + \left(3 + \frac{6}{k}\right) = 15\,. \tag{1.2}$$

Moreover, one can also consider backgrounds in which the level is not uniquely fixed by the constraint on the central charge. The simplest example is the class of vacua

$$\mathbb{R}^{1,3} \times \left[\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\right]_k \times \left[\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right]_m \tag{1.3}$$

which has $c = 15$ only if the supersymmetric level k of the cigar is given by the fraction

$$k = \frac{2m}{m+2} \qquad m = 2, 3, 4 \dots \tag{1.4}$$

The level of the SU(2)-coset is quantized, as it descends from a compact group.

These backgrounds are especially interesting in the context of non-critical holography [3, 14] in which these closed string backgrounds are dual to certain non-gravitational theories obtained by taking a double scaling limit of strings on non-compact Calabi-Yau

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manifolds of the form

$$x^2 + y^2 + u^2 + w^m = \mu\,. \tag{1.5}$$

The most well studied example is that of the deformed conifold ($m = 2$ or $k = 1$). For this case, it has been proven [15, 16] that the B-model on the deformed conifold has a worldsheet description as a twisted supercoset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ at level $k = 1$. A similar relationship for the more general models in (1.3) has also recently been discussed in [17], by relating these models to double scaling limits of certain matrix models.

Good progress in the $k = 1$ case was possible because the full set of observables in the topologically twisted coset CFT was obtained by E. Frenkel in the appendix to [15]. In this paper, we address in detail the cohomology of the cigar at any level k and exhibit the complete set of topological observables by adapting the purely algebraic techniques used in E. Frenkel's appendix to [15]. We will find that the topological observables are in one to one correspondence with the singular vectors in the Wakimoto modules constructed over the $\mathrm{SL}(2,\mathbb{R})$ current algebra. This is reminiscent of statements in earlier work on the relation between $N = 2$ topological singular vectors, and their relation to $\mathrm{SL}(2,\mathbb{R})$ singular vectors [18 − 20].

These non compact Gepner models are also intimately connected with some bosonic string theories, the most well-known example being the relationship between the $k = 1$ cigar and the $c = 1$ string [15]. More recently, a particular class of correlation functions in the (integer) level k cigar times a level k minimal model has been mapped to those in the $(k, 1)$ bosonic minimal string [21 − 23] [1]. The results we obtain should be instrumental in further clarifying the relation between topological cosets and bosonic string theories.

Organization

In section 2, we review the worldsheet formulation of the topologically twisted supercoset conformal field theory. In section 3, we compute the closed string cohomology of the twisted supercoset. We first discuss the integer level case in some detail in section 3.3 to clarify the techniques we use. The general proof for the rational level case is given in section 3.4. An appendix discusses a concrete example in detail which illustrates the abstract proof in the main text. It also contains some of the background in affine algebra that is useful to fully prove the statements in the main text.

2. The twisted cigar at level k

2.1 The $\mathcal{N} = 2$ superconformal symmetry

To begin with, we consider the $\mathcal{N} = 2$ supersymmetric topologically twisted cigar at level k. We briefly review the construction of the conformal field theory and the twisting procedure.

[1] See also [24] which relates twisted $\mathrm{AdS}_3 \times S^3$ with k units of flux through the S^3 to the bosonic $(k, 1)$ minimal string.

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The $\mathcal{N}=1$ currents of the parent SL(2, $\mathbb{R}$) theory at level k has affine currents J^a and fermions ψ^a that satisfy the OPE

$$\begin{aligned}J^a(z)\,J^b(w) &\sim \frac{g^{ab}\,k/2}{(z-w)^2}+\frac{f^{ab}_c\,J^c}{z-w}\\ J^a(z)\,\psi^b(w) &\sim \frac{if^{ab}_c\,\psi^c}{z-w}\\ \psi^a(z)\,\psi^b(w) &\sim \frac{g^{ab}}{z-w}\,.\end{aligned} \tag{2.1}$$

Our conventions are left-right symmetric, with $g_{ab}=\mathrm{diag}(+,+,-)$ and $f^{123}=\overline{f}^{123}=1$. In order to define the $\mathcal{N}=2$ currents, we first define[2]

$$j^a = J^a - \hat{J}^a = J^a - \frac{i}{2}f^a_{bc}\,\psi^b\,\psi_c\,. \tag{2.2}$$

The currents j^a commute with the free fermions ψ^a and generate a bosonic SL(2, $\mathbb{R}$) at level $k+2$. The Hilbert space of the original $\mathcal{N}=1$ SL(2, $\mathbb{R}$) model factorizes into a purely bosonic SL(2, $\mathbb{R}$) at level $k+2$ and three free fermions. We now implement the coset following Kazama-Suzuki [25]. The currents of the $\mathcal{N}=2$ superconformal algebra are:

$$\begin{aligned}T &= T_{\mathrm{SL}(2,\mathbb{R})}-T_{\mathrm{U}(1)} \qquad G^\pm = \sqrt{\frac{2}{k}}\,\psi^\pm\,j^\pm\\ J^R &= -\frac{2}{k}\,j^3+\left(1+\frac{2}{k}\right)\,\psi^+\,\psi^- = -\frac{2}{k}\,J^3+\psi^+\psi^-\end{aligned} \tag{2.3}$$

where we have defined $\psi^\pm = \frac{\psi^1\pm i\psi^2}{\sqrt{2}}$ and $j^\pm = \frac{j^1\pm ij^2}{\sqrt{2}}$ and

$$T_{\mathrm{U}(1)} = -\frac{1}{k}\,J^3J^3+\frac{1}{2}\psi^3\partial\psi^3\,. \tag{2.4}$$

One can check that these currents generate an $\mathcal{N}=2$ superconformal algebra with central charge $c=3+\frac{6}{k}$.

2.2 Gauging

The axial gauging of the coset is done by adding an extra boson X and superpartner ψ^X, and restricting to the cohomology of an additional gauging BRST charge $Q_{\mathrm{U}(1)}$, whose left-moving current is

$$J_{BRST} = C\,(J^3 - i\sqrt{\frac{k}{2}}\,\partial X) + \gamma'\,(\psi^3-\psi^X)\,. \tag{2.5}$$

Here, (B,C) is a $(1,0)$ ghost system associated with this BRST symmetry with central charge $c=-2$. The field X indicates the boson which will be identified with the angular direction of the cigar geometry [26, 27]. The β',γ' superghosts combine with the fermions ψ^3 and ψ^X to form a Kugo-Ojima quartet that decouples from the cohomology [28]. We can therefore safely ignore it in what follows.

[2] We follow the conventions in [15]. With these conventions, $j^3 = J^3+\psi^+\psi^-$.

The gauging current is defined to be

$$J_g = J^3 - i\sqrt{\frac{k}{2}}\partial X \tag{2.6}$$

From the definition, it is easy to check that it is a null-current and has non-singular operator product expansions with the energy momentum tensor T, and $\mathrm{U}(1)_R$ current J^R, and also with itself. The BRST operator $Q_{\mathrm{U}(1)}$ of the cigar theory is defined to be

$$Q_{\mathrm{U}(1)} = \oint dz\ J_{BRST}. \tag{2.7}$$

The cohomology with respect to the gauging BRST operator is standard to compute [29, 30]. The calculation essentially follows from the $Q_{\mathrm{U}(1)}$ exactness of both the total scaling operator, as well as the total charge in the gauged direction, and from the fact that these operators are diagonalizable in the Hilbert space. Those facts imply that non-trivial cohomology elements carry no oscillator excitations in the original gauged direction, nor in the auxiliary direction X. Also, the gauging locks the charge in the direction X to the charge in the gauged direction of the original model. (There is a trivial degeneracy in the zero-mode of the ghosts which we can safely ignore.) The details of this cohomological calculations are discussed in [30]. The net result, as explained, is according to intuition: excitations in the gauged directions are removed from the theory.

2.3 Wakimoto Representation of $\mathrm{SL}(2,\mathbb{R})$

In what follows, we will make good use of the Wakimoto free field representation of the $\mathrm{SL}(2,\mathbb{R})$ currents:

$$\begin{aligned} j^- &= \beta \\ j^3 &= \beta\gamma + \sqrt{\frac{k}{2}}\,\partial\phi \\ j^+ &= \beta\gamma^2 + \sqrt{2k}\,\gamma\,\partial\phi + (k+2)\,\partial\gamma. \end{aligned} \tag{2.8}$$

The energy momentum tensor in these variables is given by

$$T_{(\mathrm{SL}(2,\mathbb{R})} = \beta\,\partial\gamma - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2k}}\,\partial^2\phi - \frac{1}{2}\psi^+\partial\psi^- - \frac{1}{2}\psi^-\partial\psi^+\,. \tag{2.9}$$

We collect the various fields and some of their properties in table 1.

2.4 Twisting

We consider the topologically twisted theory whose stress tensor is defined by

$$T \longrightarrow T + \frac{1}{2}\partial J^R - \left(1 - \frac{1}{k}\right)\partial J_g \tag{2.10}$$

where J^R is the R-current that appears in (2.3). We have modified the usual definition of the twisted theory by adding a certain multiple times the gauging current J_g. As far

Field	Δ	Q	c
X	$-$	0	1
ϕ	$-$	$\sqrt{\frac{2}{k}}$	$1+\frac{6}{k}$
(ψ^+,ψ^-)	$(1/2,1/2)$	$-$	1
(β,γ)	$(1,0)$	$-$	2
(B,C)	$(1,0)$	$-$	-2

Table 1: List of fields in the untwisted theory, their conformal weight Δ, the background charge Q for the bosons and the central charge c.

as the physical obervables are concerned, this addition makes no difference, as all physical quantities are uncharged under the gauging current. However, the twist as defined in (2.10) makes contact with early attempts [32, 21] to relate the twisted coset theory to a bosonic $c<1$ string theory (i.e. $c<1$ matter coupled to bc (reparametrization) ghosts). Let us see how this comes about: using the explicit expressions for the currents in terms of the Wakimoto free fields, the twisted energy momentum tensor in (2.10) can be written as

$$T=-\partial\beta\gamma-\frac{1}{2}(\partial\phi)^2-\frac{k+1}{\sqrt{2k}}\,\partial^2\phi-\frac{1}{2}(\partial X)^2+i\frac{k-1}{\sqrt{2k}}\,\partial^2 X \\ -\frac{1}{2}\psi^+\partial\psi^- -\frac{1}{2}\psi^-\partial\psi^+ +\frac{3}{2}\partial(\psi^+\psi^-) \qquad (2.11)$$

We collect in table 2, the conformal dimensions and central charges of the twisted theory. We remark here that the precise combination of R-current and gauging current in (2.10) was chosen to get the coefficient of the last term in (2.11) to be precisely 3/2. From the table, we see that this has caused the fermions on the cigar (ψ^+,ψ^-) to have spins $(-1,2)$. Furthermore, we observe that the field content of the twisted theory is identical to the free

Field	Δ	Q	c
X	$-$	$-i\left(\sqrt{2k}-\sqrt{\frac{2}{k}}\right)$	$1-6\frac{(k-1)^2}{k}$
ϕ	$-$	$\sqrt{\frac{2}{k}}+\sqrt{2k}$	$1+6\frac{(k+1)^2}{k}$
(ψ^+,ψ^-)	$(-1,2)$	$-$	-26
(β,γ)	$(0,1)$	$-$	2
(B,C)	$(1,0)$	$-$	-2

Table 2: List of fields in the twisted theory, their conformal weight Δ, the background charge Q for the bosons and the central charge c.

field formulation of the bosonic $c<1$ string theory by identifying X with the matter field, ϕ with the Liouville field [31], and the fermions on the cigar with the (b,c) ghosts.

The above remark indicates that the results we obtain are pertinent to the tentative correspondence with bosonic string theories that was mentioned in the introduction (but we leave this question for future work). We continue to work with the $(\beta,\gamma,\phi,X,b,c)$

variables in what follows, and will always consider states in the Hilbert space

$$\mathcal{H}_{sl_2} \otimes \mathcal{H}_{b,c} \otimes \mathcal{H}_X \tag{2.12}$$

where sl_2 in the subscript refers to the free field (Wakimoto) module of the bosonic SL$(2, \mathbb{R})$ algebra at level $k+2$ in terms of the (β, γ, ϕ) variables, and $\mathcal{H}_{bc}$ and $\mathcal{H}_X$ are the usual Fock spaces of the fermions and bosons. As we will see, an important grading on states in this product Hilbert space will be provided by the fermion number (charge under the bc current) which we will henceforth refer to as the ghost number, in analogy with the bosonic string.

2.5 The total BRST operator

In the topologically twisted cigar theory, the BRST currents are given by

$$Q_{top} := Q^+ = \oint G^+ = \oint \psi^+ j^+ = \oint c\, j^+ \tag{2.13}$$

while the other twisted supercurrents are identified with

$$G^- = \psi^-\, j^- = b\, \beta \tag{2.14}$$

where we have used the Wakimoto representation for the currents, and the renaming:

$$j^- = \beta \qquad \text{and} \qquad (\psi^+, \psi^-) = (c, b)\,. \tag{2.15}$$

The quartet of the ghosts (β, γ) and (B, C) have total central charge zero.

We now restrict to the cohomology of the two BRST charges associated with the gauging and twisting, and define the operator

$$Q = Q_{top} + Q_{\mathrm{U}(1))} = \int dz \left[C \left(j^3 - i\sqrt{\frac{k}{2}}\partial X - cb \right) + cj^+ \right] \tag{2.16}$$

Thus, we have a sum of two commuting BRST operators. We already gave an intuition for the role of the BRST term associated to gauging. In the following we study the additional effect of the twisting on the set of observables, i.e. we compute the total BRST cohomology.

3. Closed string cohomology

Our primary goal in this paper will be to compute the cohomology of Q in the Wakimoto modules W_j for all positive values of the supersymmetric level k. As will become clear, the calculation will also provide the cohomology of Q in the dual Wakimoto modules, in the Verma modules, as well as the cohomology for irreducible modules. The calculation we perform is very much dependent on the submodule structure of the Wakimoto modules. In the following subsections, for the readers convenience, we will discuss cases with increasing difficulty in embedding structure, culminating in the general proof by induction for the most intricate case of rational level.

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We first collect a few general results about Wakimoto modules and their cohomology. As the explicit Wakimoto representation of the current algebra generators suggest (see equation (2.8)), the Wakimoto modules W_j are defined by embeddings of the $\mathrm{SL}(2,\mathbb{R})$ current algebra into free fields such that the modules W_j are free over the Lie subalgebra spanned by j_n^3 for $n < 0$ and j_n^- for $n \leq 0$ and co-free over the Lie subalgebra spanned by j_n^- for $n > 0$. (Co-free means that the dual representation is free with respect to this generator. Here, it is an invariant way of encoding that the operators γ_{-n} act freely as creation operators on the vacuum.)

- For $j \geq -\frac{1}{2}$, the Wakimoto modules are isomorphic to the Verma modules built on the same highest weight state while for $j \leq -\frac{1}{2}$, it is the dual W_j^* that is isomorphic to the Verma module. This was proven in [33] by explicitly comparing determinant formulae computed in the respective modules, and by constructing a bijective map. (We use here that the supersymmetric level k of the cigar is positive.)

- It should be noted then that since we can map Wakimoto modules to Verma modules, and vice versa, the results that we will derive for Wakimoto modules in the following can straightforwardly be translated into complete results for the cohomology in Verma modules as well.

- For the dual Wakimoto module, a lemma in the appendix of [15] states that

$$H^q(W_j^*) = \delta_{q,1}\{j\} \tag{3.1}$$

  where q refers to the ghost number (of the corresponding operator) mentioned earlier. The proof of this fact stated in [15] can be found by noting that the scaling operators in the (b,c) sector, as well as in the (β,γ) sectors are BRST exact. Since both operators are diagonalizable, it follows that no non-trivial excitations in these directions are allowed for cohomologically non-trivial operators. Also, no ϕ oscillators are then allowed, because of the gauging of the U(1) current. Then, in this zero-level subspace, one can explicitly find the form of the Q_{top} operator, which reduces to its zero-mode term ($Q_{top} = c_0 j_0^+$). It is then easy to show by direct calculation that the cohomology does indeed localize as claimed in the lemma in the appendix of [15], for any level k, and any dual Wakimoto module (see equation (3.1)).[3]

It turns out that the above rather straightforward calculation of a cohomology in a dual Wakimoto module, in a specific free field realization, is the only calculation that needs to be done explicitly. Indeed, we have now seen that Wakimoto modules and Verma modules are interchangeable (in the sense of [33]). Moreover, the embedding diagrams between Verma modules have been constructed in the ground breaking work [34] (see also [35]). It will turn out that the short exact sequences that follow from the embedding diagrams of Verma modules, along with the specific cohomology computed above, contain sufficient information to compute the cohomology for all Wakimoto and Verma modules.

[3] Explicitly, the one non-trivial state in the cohomology is the highest weight state of the dual Wakimoto module tensored with the down ghost vacuum.

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3.1 Singular vectors in Verma modules

The technique we use to compute the closed string cohomology relies on knowing the submodule structure of a given Wakimoto module, and hence on the structure of embeddings of singular vectors in the module. Since Wakimoto modules are isomorphic to Verma modules or their duals, we can use the results on embedding diagrams for the latter. Recall that Verma modules can occur as submodules of a Verma module when they are built on singular vectors (which by definition are annihilated by all creation operators, and therefore constitute a new highest weight state). A most useful result is then the location of the singular vectors within a given Verma module. The Kac-Kazhdan formula [36] states that a Verma module V_j of spin j over the (supersymmetric) level k $\mathrm{SL}(2,\mathbb{R})$ current algebra has singular vectors whenever the spin j satisfies, for some integer r, s:

$$2j+1 = r + ks \quad \text{such that} \quad rs > 0 \quad \text{or} \quad r > 0,\, s = 0\,. \tag{3.2}$$

For each solution, there is a singular vector labeled by the spin j'

$$j' = j - r \tag{3.3}$$

and at relative conformal dimension $\Delta h = rs$ with respect to the highest weight state in V_j. It will turn out (see subsection 3.2) that when a Verma module has only one singular vector, the calculation of the cohomology is rather straightforward. It is also clear that in the case when there are at least two solutions to the Kac-Kazhdan equation for a given value of the spin j and the level k, the level is rational. Thus, the case of irrational levels will be (implicitly) included when we discuss the general calculation for the case of a single singular vector.

3.2 Linear submodule structure

We start out our discussion with a very much restricted subset of all cases we consider, in order to connect with results already obtained in the literature. Some steps of the derivation will be postponed until the full treatment in the next subsections, in order to first motivate the use of the analysis.

For j values that satisfy $2j+1 \in k\mathbb{Z}$, the submodule structure of the Wakimoto modules is very similar to those for the level $k=1$, discussed in [15] and the singular vectors appear in a nested fashion. For $j \geq 0$, the embedding diagram for the Wakimoto module $W_j = V_j$ is given by:

$$V_j \longrightarrow V_{-j-1} \longrightarrow V_{j-k} \longrightarrow V_{k-j-1} \ldots \longrightarrow V_{-1/2} \tag{3.4}$$

where we have shown the end point $V_{-1/2}$ of the embedding diagram that always arises for j half-integer. For j integer, the end point is always $j=-1$. We will derive this embedding diagram in more detail in the next sections. Similarly, for $j<0$, the sub-module structure of the Wakimoto module W_j is as follows:

$$V_j \longleftarrow V_{-j-k-1} \longleftarrow V_{j+k} \longleftarrow \ldots \longleftarrow V_{-1/2} \tag{3.5}$$

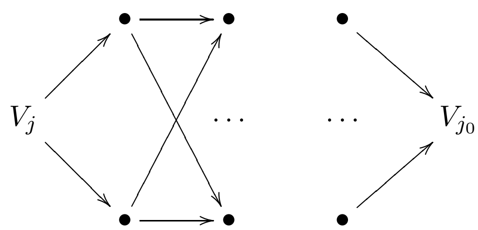
The analysis of the cohomology is exactly as in the appendix to [15] and we obtain the following result for the cohomology. (We restrict ourselves to the only non-trivial j-values, which are either half-integer or integer.) We get:

$$\begin{aligned} \text{For } j = -\frac{k}{2}, \dots, -1, -\frac{1}{2} : \quad & H^1(W_j) = \{j\} \\ \text{For } j > -\frac{1}{2} : \quad & H^1(W_j) = \{j, j-k, \dots, k-j-1, -j-1\} \quad \text{and} \\ & H^2(W_j) = \{j-k, j-2k \dots k-j-1, -j-1\} \\ \text{For } j < -\frac{k}{2} : \quad & H^1(W_j) = \{-j-k-1 \dots, j\} \quad \text{and} \\ & H^0(W_j) = \{-j-k-1, \dots, j+k\} \end{aligned} \tag{3.6}$$

Following the discussion in [37], we can infer from these spin values and the ghost number at which the cohomology arises, the values of the momentum of the auxiliary boson X, and write explicit cohomology representatives. It is easy to check that these coincide in the case at hand with the representatives constructed in [21]. It was also observed there that those representatives are valid only for $2j+1 \in k\mathbb{Z}$, i.e. the case to which we restricted above. In the coming sections, we will see that these are only a subset of all possible cohomology elements, and we now turn to finding these elements. Moreover, we give many more details of the derivation of the cohomology, and the above case will be included as a special case.

3.3 General Result for integer level

In this section, we give a proof by induction for the cohomology of the BRST operator Q in the Wakimoto modules for integer level k generalizing the examples worked out in the appendix (which can be consulted first by the reader prefering a warm-up). As we saw in the examples in the appendix, the submodule structure of W_j takes the form



(3.7)

Here, the dots in the diagram indicate singular vectors that appear in either the original module V_j, or those appearing in the submodules. Let us elaborate a bit more on how the spin j-values of the dots can be derived. For k integer, there is a simple way to generate all spins j appearing in the embedding at one go by using affine Weyl reflections about the spin values $j = -\frac{1}{2}$ and $j = -\frac{k+1}{2}$. We generate the diagram above using affine Weyl reflections and show, by analyzing the Kac-Kazhdan equation, that the j-values generated indeed correspond to all singular vectors.

By using reflections around the points $j = -\frac{1}{2}$ and $j = -\frac{k+1}{2}$, we can reflect any half-integer j spin into the range

$$-\frac{(k+1)}{2} \leq j_0 \leq -\frac{1}{2}\,. \tag{3.8}$$

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where j_0 denotes all possible end-points to an embedding diagram. First of all, we remark that all j_0 within this bound have a trivial embedding diagram, consisting of a single point (since there are no singular vectors as is easily checked from the Kac-Kazhdan equation). The embedding diagrams for other j-values can be built up inductively by adding a pair of dots at each step. The j-values of the successive dots are obtained by affine Weyl reflections about the values $j = -\frac{1}{2}$ and $j = -\frac{(k+1)}{2}$. At the first level, this leads to the j values $\{-j_0 - 1, -j_0 - k - 1\}$. Affine Weyl reflections on these dots lead to the pair $\{j_0 - k, j_0 + k\}$, $\{-j_0 - 1 + k, -j_0 - 2k - 1\}$ and so on. Thus, reading the diagram backwards, the pairs follow the pattern $\{-j_0 - 1 + nk, -j_0 - 1 - (n + 1)k\}$ followed by the pair $\{j_0 - nk, j_0 + nk\}$ and so on recursively. In this way, embedding diagrams for all half-integer j can be constructed, and the precise labeling of the dots obtained. We exhibit the embedding diagram for $j = -j_0 - 1 + nk$ as an example.

$$\begin{tikzcd}
 & V_{j_0+nk} \arrow[r] \arrow[ddr] & V_{-j_0-1+(n-1)k} & V_{j_0+k} \arrow[r] \arrow[ddr] & V_{-j_0-1} \arrow[dr] & \\
V_j \arrow[ur] \arrow[dr] & & \cdots & \cdots & & V_{j_0} \\
 & V_{j_0-nk} \arrow[r] \arrow[uur] & V_{-j_0-1-nk} & V_{j_0-k} \arrow[r] \arrow[uur] & V_{-j_0-k-1} \arrow[ur] &
\end{tikzcd} \tag{3.9}$$

We note that the case $2j + 1 \in kZ$ is a special case of the above, in which the double string embedding diagram collapses onto a single string embedding diagram. That special case was treated in subsection 3.2.

Singular vectors

From the embedding diagram, it is clear that there are four cases of j-values to be considered (for $n > 0$)

- $j_0 + nk$: As mentioned earlier, singular vectors are solutions to the Kac-Kazhdan equation (3.2), i.e. pairs of integers (r, s) such that $r + ks = 2j_0 + 2nk + 1$. Let us discuss the solution set in some detail: taking $(r, s) = (2j_0 + 2nk + 1, 0)$, we get the singular vector labeled by $j = -j_0 - 1 - nk$. Increasing s by one, we get the solution $(r, s) = (2j_0 + 1 + (2n - 1)k, 1)$ leading to the singular vector $j = -j_0 - 1 - (n - 1)k$. Proceeding this way, we find all s values from 1 up to $s = 2n - 1$, in which case we obtain the singular vector corresponding to $j = -j_0 - 1 + (n - 1)k$. If we define $S^+_{x,y} = \{j_0 + xk, j_0 + (x + 1)k, \dots, j_0 + yk\}$, and $S^-_{x,y} = \{-j_0 - 1 + xk, -j_0 - 1 + (x + 1)k, \dots, -j_0 - 1 + yk\}$, then the singular vectors corresponds to the list $S^-_{-n,n-1}$.
- $j_0 - nk$: A similar analysis leads to the set of singular vectors identical to the case above, and we get the list of singular vectors $S^-_{-n,n-1}$.
- $-j_0 - 1 + nk$: We now get singular vectors of the form $j_0 \pm mk$ with $0 \leq m \leq n$, which corresponds to the list $S^+_{-n,n}$.
- $-j_0 - 1 - nk$: We get the list of singular vectors $S^+_{-n,n}$. We have thus verified that the embedding diagram of V_{-j_0-1+nk} indeed coincides with (3.9).

Proof by induction

We now claim that the cohomology of the Wakimoto module is given in terms of the singular vectors appearing in the module. We will give the details of the proof in the case $j = -j_0 - 1 + nk$. The other cases can be proven similarly. We will prove this claim by induction, by assuming that the result holds for all $m \leq n-1$ and then show that the result also holds for $m = n$. For $j = -j_0 - 1 + nk$, the embedding diagram is shown in (3.9). In this case, the following short exact sequences are valid:

$$0 \longrightarrow \mathrm{Im}(V_{-j_0-1+(n-1)k} \oplus V_{-j_0-1-nk}) \longrightarrow V_{j_0+nk} \oplus V_{j_0-nk} \longrightarrow \mathrm{Im}(V_{j_0+nk} \oplus V_{j_0-nk}) \longrightarrow 0 \tag{3.10}$$

$$0 \longrightarrow \mathrm{Im}(V_{j_0+nk} \oplus V_{j_0-nk}) \longrightarrow V_{-j_0-1+nk} \longrightarrow I_{-j_0-1+nk} \longrightarrow 0 \tag{3.11}$$

where I_j refers to the irreducible module obtained by modding out the singular vectors in V_j. The second sequence indicates that when we mod out by the leading singular vectors, the Verma module becomes irreducible, while the first sequence quantifies the fact that in that modding, we have double counted the submodules common to the two leading singular vectors. Now, we observe that in order to compute the cohomology using induction, we also need to postulate the cohomology of the various images of direct sums of Verma modules in other Verma modules. We postulate the following ansatz for the Wakimoto modules

$$\begin{aligned} H^1(W_{-j_0-1\pm mk}) &= S^+_{-m,m} \cup \{-j_0 - 1 \pm mk\} \\ H^2(W_{-j_0-1+mk}) = H^0(W_{-j_0-1-mk}) &= S^+_{-m,m} \\ H^1(W_{j_0\pm mk}) &= S^-_{-m,m-1} \cup \{j_0 \pm mk\} \\ H^2(W_{j_0+mk}) = H^0(W_{j_0-mk}) &= S^-_{-m,m-1} \end{aligned} \tag{3.12}$$

and introduce the following ansatz for the cohomology of the image modules

$$\begin{aligned} H^1(\mathrm{Im}(V_{-j_0-1+(n-1)k} \oplus V_{-j_0-1-nk})) &= \text{singular vectors in } V_{j_0\pm nk} \\ &= S^-_{-n,n-1} \end{aligned} \tag{3.13}$$

$$\begin{aligned} H^2(\mathrm{Im}(V_{-j_0-1+(n-1)k} \oplus V_{-j_0-1-nk})) &= \text{singular vectors in } V_{-j_0-1+(n-1)k} \\ &= S^+_{-(n-1),(n-1)} \end{aligned} \tag{3.14}$$

where in the right hand side of (3.14), we could have used V_{-j_0-1-nk} as well. Our ansatz can be explained as follows: the ghost number one cohomology H^1 of the image of the sum of two Verma modules $V_{j_1} \oplus V_{j_2}$ is given by the set of singular vectors in the Verma module V_j in which j_1 and j_2 are the leading singular vectors. Similarly, cohomology at ghost number two H^2 of the image is given by the singular vectors in either V_{j_1} or V_{j_2}, which coincide in all cases we consider. We will see that this ansatz generalizes to the rational level case also.

We now have all the ingredients to complete our proof by induction. Let us consider the short exact sequence in (3.10). By the induction hypothesis, we know the cohomology of the first and middle element in the sequence. Considering the long exact sequence of cohomology, and using the induction hypothesis, the long exact sequence collapses to

$$\begin{aligned}0 \longrightarrow S^{-}_{-n,n-1} \longrightarrow S^{-}_{-n,n-1} \cup \{j_0 - nk, j_0 + nk\} \longrightarrow H^1(\mathrm{Im}(V_{j_0+nk} \oplus V_{j_0-nk})) \longrightarrow \\ \longrightarrow S^{+}_{-(n-1),n-1} \longrightarrow S^{-}_{-n,n-1} \longrightarrow H^2(\mathrm{Im}(V_{j_0+nk} \oplus V_{j_0-nk})) \longrightarrow 0 \quad (3.15)\end{aligned}$$

Solving for the cohomology, we get $H^0(\mathrm{Im}(V_{j_0+nk} \oplus V_{j_0-nk})) = 0$,

$$\begin{aligned}H^1(\mathrm{Im}(V_{j_0+nk} \oplus V_{j_0-nk})) &= S^{+}_{-n,n} \\ &= \text{singular vectors in } V_{-j_0-1+nk} \quad (3.16)\\ H^2(\mathrm{Im}(V_{j_0+nk} \oplus V_{j_0-nk})) &= S^{-}_{-n,n-1} \\ &= \text{singular vectors in } V_{j_0\pm nk} \quad (3.17)\end{aligned}$$

where in (3.16) and (3.17), we indicate that the result is what one would expect, given our ansatz for the cohomology of the image modules in (3.13) and (3.14). We also find that the cohomology of the dual of the image is exactly the same as this one, with the exchange of ghost number zero H^0 and ghost number two H^2 cohomology. Let us now consider the short exact sequence dual to (3.11):

$$0 \longrightarrow I_{-j_0-1+nk} \longrightarrow V^{*}_{-j_0-1+nk} \longrightarrow \mathrm{Im}^{*}(V_{j_0+nk} \oplus V_{j_0-nk}) \to 0 \quad (3.18)$$

Note that although we have taken the dual sequence, the irreducible module has remained unchanged, since it is canonically isomorphic to its dual. The long exact sequence corresponding to (3.18) now takes the form

$$\begin{aligned}0 \longrightarrow H^0(I_{-j_0-1+nk}) \longrightarrow 0 \longrightarrow S^{-}_{-n,n-1} \longrightarrow H^1(I_{-j_0-1+nk}) \longrightarrow \\ \longrightarrow \{-j_0 - 1 + nk\} \longrightarrow S^{+}_{-n,n} \longrightarrow H^2(I_{-j_0-1+nk}) \longrightarrow 0 \quad (3.19)\end{aligned}$$

which leads to

$$\begin{aligned}H^1(I_{-j_0-1+nk}) &= S^{-}_{-n,n} \\ H^2(I_{-j_0-1+nk}) &= S^{+}_{-n,n}\,. \quad (3.20)\end{aligned}$$

We now input all these results into the long exact sequence corresponding to the short sequence (3.11), and consequently it collapses to the following short sequence

$$\begin{aligned}0 \longrightarrow S^{+}_{-n,n} \longrightarrow H^1(V_{-j_0-1+nk}) \longrightarrow S^{-}_{-n,n} \longrightarrow S^{-}_{-n,n-1} \longrightarrow \\ \longrightarrow H^2(V_{-j_0-1+nk}) \longrightarrow S^{+}_{-n,n} \longrightarrow 0 \quad (3.21)\end{aligned}$$

Thus, we finally get the non-zero cohomologies

$$\begin{aligned}H^1(W_{-j_0-1+nk}) &= S^{+}_{-n,n} \cup \{-j_0 - 1 + nk\} \\ H^2(W_{-j_0-1+nk}) &= S^{+}_{-n,n} \quad (3.22)\end{aligned}$$

which agrees with the ansatz in (3.12) for $m = n$. Thus, we have proved the induction hypothesis for one case $j = -j_0 - 1 + nk$, and all other cases can be computed similarly. Note that we have also proven that the cohomologies localize on singular vectors (and precisely how) in the case of integer level and for any spin. We now turn to the most intricate case of fractional level k.

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3.4 The cohomology for positive rational level

We consider the fractional level $k = \frac{p}{q}$, where p, q are mutually prime strictly positive integers. We already argued that this is going to be the most non-trivial case, since two or more singular vectors in a given Verma module imply that k is fractional. Moreover, we note that for fractional k, solutions to the Kac-Kazhdan equations will only occur for the set of spins that are multiples of $\frac{1}{2q}$. We therefore from now on restrict to this non-trivial subset (since all other cases can be treated more straightforwardly).

Embedding diagrams

For this case as well, we can algorithmically construct the embedding diagrams for all Verma modules, using the results of Malikov [34] (see also [35]). The diagrams have the same "double-string" shape as before, but are now generated by ρ-centered affine Weyl reflections that depend on the particular spin under study. We discuss in detail in the appendix A how these embedding diagrams are constructed. Although this is the most systematic and all-at-once procedure, below, we will follow a more pedestrian route. We will reconstruct the embedding diagrams inductively simply by locating the singular vectors in a given module. Although this approach may be more accessible and constructive, its a posteriori justification lies in its agreement with the proof for the embedding diagram given in [34].

The cohomology for a simple case

To compute the cohomology, we follow an inductive procedure. First of all, when $2j+1$ is not of the form

$$2j+1 = r + ks \quad \text{such that} \quad rs > 0 \quad \text{or} \quad r > 0, \;\; s = 0\,, \tag{3.23}$$

then the module has no singular vectors and is irreducible. We have then that the Verma module is equivalent to its dual, and is equivalent to the Wakimoto module and its dual. This is because for irreducible Verma modules, there exists a non-degenerate Shapolov form which can be used to construct a canonical isomorphism between the Verma module and its dual. This also implies that the cohomology of an irreducible module is the same as that of its dual. Thus all these modules have cohomology $H^n(V_j = W_j = W_j^*) = \delta_{1,n}\{j\}$. This completes the calculation of the cohomology in the case of irreducible modules.

The cohomology in the case of one singular vector

Next, we move on to the case where the Verma module V_{j_1} under consideration has precisely one singular vector. The values of j_1 for which this is the case can be parameterized by the corresponding values of (r_1, s_1), which are related to j_1 by

$$2j_1 + 1 = r_1 + ks_1\,. \tag{3.24}$$

When $2j_1+1$ is positive, the couple (r_1, s_1) satisfies $r_1 \in \{1, 2, \ldots, p\}$ and $s_1 \in \{0, 1, \ldots, q-1\}$. When $2j_1+1$ is negative, we have that $r_1 \in \{-1, -2, \ldots, -p\}$ and $s_1 \in \{-1, -2, \ldots, -q\}$. We have thus $2.p.q$ values of j_1 with precisely one singular vector.

The singular vectors are at values of the spin $j_1^{(1)}$ which are given by $2j_1^{(1)} + 1 = 2j_1 + 1 - 2r_1 = -r_1 + ks_1$. (Note that this expression can be of either sign.) It is now not too difficult to prove that none of the Verma modules associated to the singular vectors can have any singular vectors themselves. Thus, they are irreducible, and we have already computed their cohomology. We can then proceed to compute the cohomology for the case of one singular vector. We have the exact sequences:

$$0 \longrightarrow V_{j_1^{(1)}} \longrightarrow V_{j_1} \longrightarrow I_{j_1} \longrightarrow 0$$
$$0 \longrightarrow I_{j_1} \longrightarrow V_{j_1}^* \longrightarrow V_{j_1^{(1)}} \longrightarrow 0\,.$$

The first sequence indicates that modding out by the single singular vector gives an irreducible module, while the second follows by dualizing and from the fact that the modules I_{j_1} and $V_{j_1^{(1)}}$ are irreducible. Suppose that j_1 is positive; then $V_{j_1} = W_{j_1}$. Then we first use the second exact sequence to compute the cohomology of I_{j_1}:

$$0 \longrightarrow H^1(I_{j_1}) \longrightarrow \{j_1\} \longrightarrow \{j_1^{(1)}\} \longrightarrow H^2(I_{j_1}) \longrightarrow 0\,. \tag{3.25}$$

We thus have that

$$H^1(I_{j_1}) = \{j_1\} \qquad H^2(I_{j_1}) = \{j_1^{(1)}\}. \tag{3.26}$$

We can feed this information into the first exact sequence and we obtain:

$$0 \longrightarrow \{j_1^{(1)}\} \longrightarrow H^1(W_{j_1}) \longrightarrow \{j_1\} \longrightarrow 0 \longrightarrow H^2(W_{j_1}) \longrightarrow \{j_1^{(1)}\} \longrightarrow 0 \tag{3.27}$$

which gives us the result:

$$H^1(W_{j_1}) = \{j_1\} \cup \{j_1^{(1)}\} \qquad H^2(W_{j_1}) = \{j_1^{(1)}\}\,. \tag{3.28}$$

These are the set of the parent spin and the singular vector, and the singular vector respectively.

For j_1 negative, we have that the Verma module is equivalent to the dual Wakimoto module, and repeating the analysis above, we find that

$$H^1(W_{j_1}) = \{j_1\} \cup \{j_1^{(1)}\} \qquad H^0(W_{j_1}) = \{j_1^{(1)}\}\,. \tag{3.29}$$

Thus we have computed the cohomologies for all modules with a single singular vector.

The location of singular vectors in Verma modules

We now turn to a classification of the cases where the parent module has precisely n singular vectors. It is not too difficult to see that these cases are described as follows. Define the spin j_n to be given by:

$$2j_n + 1 = r_n + ks_n\,. \tag{3.30}$$

For positive spin, we have that j_n lies in the set corresponding to $r_n \in \{(n-1)p+1, (n-1)p+2, \ldots, np\}$ and $s_n \in \{0, 1, \ldots, q-1\}$. When $2j_n + 1$ is negative, we have that

$r_n \in \{-(n-1)p-1, -(n-1)p-2, \ldots, -np\}$ and $s_n \in \{-1,-2,\ldots,-q\}$. We thus always have $2.p.q$ values of j_n with precisely n singular vectors. In the case of positive spin, which will take to be the case for our calculations, we have that the singular vectors correspond to the pairs $(r,s) \in \{(r_n, s_n), (r_n - p, s_n + q), \ldots, (r_n - (n-1)p, s_n + (n-1)q)\}$.

We also have a claim that we will treat in more detail. Suppose we consider the module V_{j_n}, and label the leading singular vectors in them as $V_{j^{(1)}_{n-1}}$ and $V_{j^{(n)}_{n-1}}$ [4]. Now list the singular vectors in these modules. We will find below that these are in fact, the same list of vectors. Once again, consider the leading singular vectors in this list of $n-1$ singular vectors, and label them $V^{(1,1)}_{j_{n-2}}$ and $V^{(1,n-1)}_{j_{n-2}}$. Consider the singular vectors in these modules; we claim (and prove below) that the singular vectors in either of these modules make up the remaining $n-2$ singular vectors in the original module V_{j_n}. This is key to writing down short exact sequences of Verma modules that will lead to computing cohomology.

Let's prove these claims, and start with a spin j_n module, associated to a module with n singular vectors. We have that $2j_n + 1 = r_n + ks_n$ where (r_n, s_n) are taken in the set described above (and we take them to be positive for simplicity). The resulting spins of the modules of the singular vectors are ($l_n = 1, 2, \ldots, n$):

$$2j^{(l_n)}_{n-1} + 1 = -r_n + ks_n + 2(l_n - 1)p. \tag{3.31}$$

In particular, we have the spins which we believe to be the nearest to j_n in the embedding diagram of V_{j_n}, namely those associated to $l_n = 1, n$:

$$\begin{aligned} 2j^{(1)}_{n-1} + 1 &= -r_n + ks_n = r^{(1)}_{n-1} + ks^{(1)}_{n-1} \\ 2j^{(n)}_{n-1} + 1 &= -r_n + ks_n + 2(n-1)p = r^{(n)}_{n-1} + ks^{(n)}_{n-1}. \end{aligned} \tag{3.32}$$

In the case of integer level, these appeared at an equal distance from j_n but in the rational level, this need not be the case. Modules associated to each of these j values have $n-1$ singular vectors. Indeed, the associated pairs $(r^{(1)}_{n-1}, s^{(1)}_{n-1})$ and $(r^{(n)}_{n-1}, s^{(n)}_{n-1})$ are given by the formulae

$$\begin{aligned} r^{(1)}_{n-1} &= p - r_n \in \{-(n-2)p, \ldots, -(n-1)p\} \\ s^{(1)}_{n-1} &= s_n - q \in \{-q, \ldots, -1\} \\ r^{(n)}_{n-1} &= 2(n-1)p - r_n \in \{(n-2)p, \ldots, (n-1)p\} \\ s^{(n)}_{n-1} &= s_n \in \{1, \ldots, q\}\,. \end{aligned} \tag{3.33}$$

Their ranges have been explicitly shown to make it clear that they indeed have $n-1$ singular vectors each. Let us now list the set of singular vectors in each of the modules

[4] The notation is such that the subscript tells us the number of singular vectors in a given module, while the superscripts keeps track of its line of descent from V_{j_n}.

$V_{j^{(1)}_{n-1}}$ and $V_{j^{(n)}_{n-1}}$. We will denote these respectively as $j_{n-2}^{(1,l^{(1)}_{n-1})}$ and $j_{n-2}^{(n,l^{(n)}_{n-1})}$. These satisfy

$$\begin{aligned} 2j_{n-2}^{(1,l^{(1)}_{n-1})}+1 &= -r_n^{(1)}+ks_n^{(1)}-2(l^{(1)}_{n-1}-1)p \quad \text{with} \quad l^{(1)}_{n-1}\in\{1\dots n-1\} \\ &= r_n-p+k(s_n-q)-2(l^{(1)}_{n-1}-1)p \\ &= r_n+ks_n-2l^{(1)}_{n-1}p \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} 2j_{n-2}^{(n,l^{(n)}_{(n-1)})}+1 &= -r_{n-1}^{(n)}+ks_{n-1}^{(n)}+2(l^{(n)}_{n-1}-1)p \quad \text{with} \quad l^{(n)}_{n-1}\in\{1\dots n-1\} \\ &= r_n-2(n-1)p+ks_n+2(l^{(n)}_{n-1}-1)p \\ &= r_n+ks_n-2(n-l^{(n)}_{n-1})p \end{aligned} \tag{3.35}$$

It is clear from (3.35) that the singular vectors in each of the two lists is identical as each of the l_{n-1}'s go from 1 to $n-1$. In particular, we see that

$$j_{n-2}^{(1,l^{(1)}_{n-1})}=j_{n-2}^{(n,n-l^{(n)}_{(n-1)})}\,. \tag{3.36}$$

Without loss of generality, we may therefore safely omit the first superscript and denote them as $j^{(1)}_{n-2}\dots j^{(n-1)}_{n-2}$. Let us focus on either of the leading singular vectors, which have $l_{n-1}=1,n-1$. These have the r_{n-2} and s_{n-2} values given by

$$\begin{aligned} r^{(1)}_{n-2} &= r_n-2p\in\{(n-3)p,\dots,(n-2)p\} \\ s^{(1)}_{n-2} &= s_n\in\{1,\dots,q\} \\ r^{(n-1)}_{n-2} &= r_n-(2n-3)p\in\{-(n-2)p,\dots,-(n-3)p\} \\ s^{(n-1)}_{n-2} &= s_n-q\in\{-q,\dots,-1\}\,. \end{aligned} \tag{3.37}$$

It is clear that their (r_{n-2},s_{n-2}) values are such that the modules $V_{j^{(1)}_{n-2}}$ and $V_{j^{(n-1)}_{n-2}}$ have $n-2$ singular vectors each, which we label $j_{n-3}^{(1,m^{(1)}_{n-2})}$ and $j_{n-3}^{(n-1,m^{(n-1)}_{n-2})}$ respectively. These satisfy the equations

$$\begin{aligned} 2j_{n-3}^{(1,m^{(1)}_{n-2})}+1 &= -r^{(1)}_{n-2}+ks^{(1)}_{n-2}+2(m^{(1)}_{n-2}-1)p \\ &= -(r_n-2p)+ks_n+2(m^{(1)}_{n-2}-1)p \\ &= -r_n+ks_n+2m^{(1)}p \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} 2j_{n-3}^{(n-1,m^{(n-1)}_{n-2})}+1 &= -r^{(n-1)}_{n-2}+ks^{(n-1)}_{n-2}-2(m^{(n-1)}_{n-2}-1)p \\ &= (2n-3)p-r_n+ks_n-p-2(m^{(n-1)}_{n-2}-1)p \\ &= -r_n+ks_n+2((n-1)-m^{(n-1)}_{n-2})\,. \end{aligned} \tag{3.39}$$

where $m^{(1)}_{n-2}$ and $m^{(n-1)}_{n-2}$ take values in $\{1, \ldots, n-2\}$. Once again, we see that both lists of singular vectors coincide if we identify

$$j_{n-3}^{(1,m^{(1)}_{n-2})} = j_{n-3}^{(n-1,(n-1)-m^{(n-1)}_{n-2})} \,. \tag{3.40}$$

Once again, the first superscript index is redundant and we may denote them as $j^{(1)}_{n-3} \ldots j^{(n-2)}_{n-3}$. Moreover, by comparing with (3.31), we observe that the singular vectors in both $V_{j^{(1)}_{n-2}}$ and $V_{j^{(n-1)}_{n-2}}$ (the modules corresponding to the leading second generation singular vectors) precisely account for all but two of the singular vectors in the original module V_{j_n}. We have thus proved the assertion made at the beginning of this section.

Proof by induction

We now make an educated guess for the embedding diagram of these modules. As already remarked, the guess is exact, as proven in [34] (see also our appendix A)[5]:

$$\begin{tikzcd}
 & V_{j^{(1)}_{n-1}} \arrow[r] \arrow[dr] & V_{j^{(n-1)}_{n-2}} & & V_{j^{(1)}_{2}} \arrow[r] \arrow[dr] & V_{j^{(2)}_{1}} \arrow[dr] & \\
V_{j_n} \arrow[ur] \arrow[dr] & & & \cdots \quad \cdots & & & V_{j_0} \\
 & V_{j^{(n)}_{n-1}} \arrow[r] \arrow[ur] & V_{j^{(1)}_{n-2}} & & V_{j^{(3)}_{2}} \arrow[r] \arrow[ur] & V_{j^{(1)}_{1}} \arrow[ur] &
\end{tikzcd} \tag{3.41}$$

The knowledge of the embedding diagram allows us to write down the exact sequences:

$$0 \longrightarrow \mathrm{Im}(V_{j^{(1)}_{n-2}} \oplus V_{j^{(n-1)}_{n-2}}) \longrightarrow V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}} \longrightarrow \mathrm{Im}(V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}}) \longrightarrow 0 \tag{3.42}$$

$$0 \longrightarrow \mathrm{Im}(V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}}) \longrightarrow V_{j_n} \longrightarrow I_{j_n} \longrightarrow 0 \tag{3.43}$$

The second sequence codes the irreducibility of the module obtained by modding out by the leading singular vectors, while the first one encodes the double counting in this modding out. From here on, the proof for the cohomology of the BRST operator Q is identical to the integer level case and we once again find that cohomology elements are given by the singular vectors. As for the integer level case, we need to postulate not only the cohomology of the Wakimoto modules but also those of the image modules appearing in (3.42). We make the following induction hypothesis for all m values in the range $0 \leq m \leq n-1$:

$$\begin{aligned}
\text{For } j_m > -\tfrac{1}{2} \qquad & H^1(W_{j_m} = V_{j_m}) = \{j^{(1)}_{m-1} \ldots j^{(m)}_{m-1}\} \cup \{j_m\} \\
& H^2(W_{j_m} = V_{j_m}) = \{j^{(1)}_{m-1} \ldots j^{(m)}_{m-1}\} \\
\text{For } j_m < -\tfrac{1}{2} \qquad & H^1(W_{j_m} = V^*_{j_m}) = \{j^{(1)}_{m-1} \ldots j^{(m)}_{m-1}\} \cup \{j_m\} \\
& H^0(W_{j_m} = V^*_{j_m}) = \{j^{(1)}_{m-1} \ldots j^{(m)}_{m-1}\} \\
& H^q(W^*_{j_m}) = \delta_{q,1} \; \{j_m\} \quad \forall j_m
\end{aligned} \tag{3.44}$$

[5]Again, in exceptional cases, linear embedding diagrams arise. The calculation of the cohomology is then easily adapted from the appendix to [15] and our subsection on the linear case for integer k.

while for the image modules appearing in (3.42) and (3.43), we postulate

$$\begin{aligned} H^1(\mathrm{Im}(V_{j^{(1)}_{n-2}} \oplus V_{j^{(n-1)}_{n-2}})) &= \{j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\} \\ H^2(\mathrm{Im}(V_{j^{(1)}_{n-2}} \oplus V_{j^{(n-1)}_{n-2}})) &= \{j^{(1)}_{n-3} \dots j^{(n-2)}_{n-3}\} \\ &= \{j^{(2)}_{n-1} \dots j^{(n-1)}_{n-1}\} \end{aligned} \tag{3.45}$$

where we have generalized (3.13) and (3.14). We will try to prove (3.44) for $m = n$. We begin by using our induction hypothesis in the long exact sequence corresponding to the short sequence in equation (3.42). We obtain the sequence

$$\begin{aligned} 0 \longrightarrow H^0(V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}}) \longrightarrow \{j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\} \longrightarrow \{j^{(1)}_{n-1}, j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\} \longrightarrow \\ \longrightarrow H^1(\mathrm{Im}(V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}}) \longrightarrow \{j^{(2)}_{n-1} \dots j^{(n-1)}_{n-1}\} \longrightarrow H^2(V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}}) \longrightarrow 0 \end{aligned} \tag{3.46}$$

from which we solve for the cohomology of the image modules:

$$\begin{aligned} H^1(\mathrm{Im}(V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}}) &= \{j^{(1)}_{n-1} \dots j^{(n)}_{n-1}\} \\ H^2(\mathrm{Im}(V_{j^{(1)}_{n-1}} \oplus V_{j^{(n)}_{n-1}}) &= \{j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\}. \end{aligned} \tag{3.47}$$

Thus, we have inductively verified the ansatz for the cohomology of the image modules. The cohomology for the dual of the image is computed similarly, and one obtains the same cohomology with the ghost number zero H^0 and ghost number two cohomology H^2 interchanged. We now plug this result into the long exact sequence corresponding to the dual of the short exact sequence in (3.43) to get

$$\begin{aligned} 0 \longrightarrow H^0(I_{j_n}) \longrightarrow 0 \longrightarrow \{j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\} \longrightarrow H^1(I_{j_n}) \longrightarrow \\ \longrightarrow \{j_n\} \longrightarrow \{j^{(1)}_{n-1} \dots j^{(n)}_{n-1}\} \longrightarrow H^2(I_{j_n}) \longrightarrow 0 \end{aligned} \tag{3.48}$$

leading to

$$\begin{aligned} H^1(I_{j_n}) &= \{j_n\} \cup \{j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\} \\ H^2(I_{j_n}) &= \{j^{(1)}_{n-1} \dots j^{(n)}_{n-1}\}\,. \end{aligned} \tag{3.49}$$

Substituting all these results into the long exact sequence corresponding to (3.43), we find

$$\begin{aligned} 0 \longrightarrow H^0(W_{j_n}) \longrightarrow 0 \longrightarrow \{j^{(1)}_{n-1} \dots j^{(n)}_{n-1}\} \longrightarrow H^1(W_{j_n}) \longrightarrow \{j_n, j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\} \longrightarrow \\ \longrightarrow \{j^{(1)}_{n-2} \dots j^{(n-1)}_{n-2}\} \longrightarrow H^2(W_{j_n}) \longrightarrow \{j^{(1)}_{n-1} \dots j^{(n)}_{n-1}\} \longrightarrow 0 \end{aligned} \tag{3.50}$$

from which we conclude

$$\begin{aligned} H^1(W_{j_n}) &= \{j_n\} \cup \{j^{(1)}_{n-1} \dots j^{(n)}_{n-1}\} \\ H^2(W_{j_n}) &= \{j^{(1)}_{n-1} \dots j^{(n)}_{n-1}\} \end{aligned} \tag{3.51}$$

By comparing with equation (3.44), we observe that we have derived the result for W_{j_n} by assuming the result for all $m < n$. This is our answer for the cohomology of Wakimoto modules and thus the complete set of topological observables for the level $k = \frac{p}{q}$ twisted $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ model.

3.5 Alternative proofs

An alternative inductive proof for the cohomology of the Wakimoto modules is possible, based on another pair of short exact sequences[6]. Again, for simplicity only, we restrict to the case of positive $2j_n + 1 = r_n + ks_n$. The induction hypothesis consists of the validity of the cohomology that we found above for modules with less than n singular vectors in (3.44), and, of the cohomology of the corresponding irreducible modules. (The latter cohomology depends on the sign of the spin.) Now, we can construct a first short exact sequence by modding out the Verma module of spin j_n by one of its two leading singular vectors, say $j_{n-1}^{(1)}$, to produce a new module X_n:

$$\begin{aligned} &0 \to V_{j_{n-1}^{(1)}} \to V_{j_n} \to X_n \to 0 \\ &0 \to I_{j_{n-1}^{(n)}} \to X_n \to I_{j_n} \to 0. \end{aligned} \tag{3.52}$$

We then used the module X_n in a second sequence which says that after eliminating the further singular vector $j_{n-1}^{(n)}$ (and none of its descendants since they have already been modded out), we obtain an irreducible module. Using these two short exact sequences and their duals it is not hard to show that the induction step can be proven (both on the cohomology of the irreducible module and on the cohomology of the Wakimoto modules). For the reader's convenience we record an intermediate result, namely the cohomology of the modules X_n (for j_n positive):

$$\begin{aligned} H^1(X_n) &= \{j_n\} \cup \{j_{n-2}^{(1)}, \ldots, j_{n-2}^{(n-1)}\} \\ H^2(X_n) &= \{j_{n-1}^{(1)}\} \cup \{j_{n-2}^{(1)}, \ldots, j_{n-2}^{(n-1)}\}\,. \end{aligned} \tag{3.53}$$

Furthermore, from the first alternative proof follows a second. We could choose to mod out first by the other leading singular vector. We obtain the short exact sequences:

$$\begin{aligned} &0 \to V_{j_{n-1}^{(n)}} \to V_{j_n} \to Y_n \to 0 \\ &0 \to I_{j_{n-1}^{(1)}} \to X_n \to I_{j_n} \to 0. \end{aligned} \tag{3.54}$$

The proof follows familiar lines. We quote the intermediate result for the cohomology of the new modules Y_n:

$$\begin{aligned} H^0(Y_n) &= \{j_{n-2}^{(1)}, \ldots, j_{n-2}^{(n-1)}\} \\ H^1(Y_n) &= \{j_n\} \cup \{j_{n-1}^{(1)}, \ldots, j_{n-1}^{(n-1)}\} \cup \{j_{n-2}^{(1)}, \ldots, j_{n-2}^{(n-1)}\} \\ H^2(Y_n) &= \{j_{n-1}^{(1)}, \ldots, j_{n-1}^{(n)}\}. \end{aligned} \tag{3.55}$$

Note that these modules have cohomology at three different ghost numbers.[7]

[6] We would like to thank Edward Frenkel for indicating to us the idea behind this alternative proof.

[7] Algebraically, it is intuitive that the two leading singular vectors give rise to proofs with slightly different structure, since only one of the two leading singular spins is related to the parent spin by an elementary Weyl reflection, as defined and discussed in the appendix A.

4. Concluding remarks

Our main result is that the cohomology of the topologically twisted coset theory is given in terms of the singular vectors of the Wakimoto modules constructed over the $SL(2,\mathbb{R})$ current algebra. The precise form in which this is realized can be seen in equation (3.44). In the process we also computed the cohomology on the dual Wakimoto modules, the Verma modules as well as the irreducible modules. We conclude by making a few remarks on further avenues to explore, and on the uses of the rather technical results we have obtained regarding the topological observables of the cigar.

- It is possible to give explicit representatives of the cohomology, using the formulae obtained for singular vectors in $SL(2,\mathbb{R})$ modules [38, 39]. Following the analysis in [15], rewriting these cohomology elements in Wakimoto variables could be very useful in establishing the relation between twisted cosets and bosonic string theories.

- It would be instructive to further analyze, as was done in [37] for the particular case of the topological cigar at level $k = 1$ (i.e. the conifold), how the operation of spectral flow acts on the observables of the topological cigar.

- The relationship between the statement that the cohomology localizes on singular vectors and the statement [20] that $N = 2$ topological singular vectors always arise (through the Kazama-Suzuki map) from $SL(2,\mathbb{R})$ singular vectors are tantalizingly close. It is fairly clear that clarifying the details of the relation will also illuminate the role of spectral flow.

- An important application we have in mind for these results is in the context of the $N = 2$ topological string (see e.g. [40] for a review). Now that we obtained the observables on the topological cigar, the algorithm to obtain the closed string spectrum of topological strings on non-compact Gepner models consisting of (twisted) $N = 2$ minimal models and topological cigars is clear. Indeed, the chiral cohomology on the $N = 2$ minimal models is straightforward to compute, and is well-known. (When working with a unitary spectrum, the powerful results of [41] make the calculation much easier.) Furthermore, one could then combine left- and right-movers to form a consistent spectrum, following Gepner's technique of GSO projecting onto integer $U(1)_R$ charges. In the present context, the GSO projection will boil down to a simple orbifold by an abelian group that will depend only on the levels involved in the non-compact Gepner model. (This is clear from the form of the $N = 2$ $U(1)_R$ current in these models and is for instance discussed in [9–13], following Gepner.) It will then be very interesting to combine these observations together with the fact that the $N = 2$ topological string basically corresponds to a bosonic string theory (see e.g. [40]). This would be a worldsheet approach to isolate integrable subsectors of string theories on (non-compact) Calabi-Yau manifolds [42].

- Since our results apply to individual factors of the Hilbert space of the supercoset, we can compute the cohomology of any linear sum of Wakimoto, Verma or irreducible modules. In particular, it will be interesting to apply this to the physical string Hilbert space, which can be based on Verma modules (equivalent to Wakimoto modules), as is the case for the parent theory (see e.g. [44]). Also, one should be able to represent the physical string Hilbert space in terms of irreducible modules, to which our results equally apply. (See e.g. the papers [45, 11, 12] for results on the physical string partition function on the coset.) Moreover, it should be remarked that the Hilbert space relevant to the topological string is generically different. Indeed, there are observables in the topological string theory that are not observables in the physical string theory (for example because they are non-normalizable). Note for instance that the Hilbert spaces used in [15, 21] to match the cohomology onto a bosonic string Hilbert space do not coincide with the physical string Hilbert space. This issue deserves further study.

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A. A few notes on affine algebra

The calculation of the cohomology performed in the bulk of the paper heavily leans on an understanding of the embedding structure of Verma modules for the rank two Kac-Moody algebra sl_2. In this appendix, we briefly review some of the mathematics involved in understanding the embedding diagrams. We hope this will make the relevant mathematics literature more readable to the interested physicist.

We introduce some notation and refer to standard text books on affine algebras for further information on the following standard concepts. We introduce the algebra sl_2, where the notation is meant to indicate that we consider the algebra over the complex numbers. We follow (in this appendix only) $su(2)$ conventions for the structure constants and metric of the corresponding affine algebra (see e.g. [43] for standard conventions). The affine algebra has simple positive roots α_1 (the simple positive root of the ordinary, horizontal $su(2)$ subalgebra) and $\alpha_0 = \delta - \alpha_1$, where δ is the imaginary root. We take the algebra to be at (bosonic) level $k_{su(2)}$.

We can define ρ-centered affine Weyl reflections (where ρ is defined in the usual way, as having inner product equal to one with all positive simple roots). These ρ-centered affine Weyl reflections s^{ρ}_{α} with respect to the root α are given in terms of the affine Weyl reflections s_α by the formula:

$$s_\alpha \hat{\lambda} = \hat{\lambda} - \frac{2(\alpha,\hat{\lambda})}{(\alpha,\alpha)}$$
$$s^\rho_\alpha(\lambda) = s_\alpha(\lambda+\rho) - \rho. \tag{A.1}$$

We will mostly refer only to the action of the ρ-centered affine Weyl reflections on the part of the weight that lies in the direction of the horizontal $su(2)$ subalgebra of the full Kac-Moody algebra (i.e. the standard spin of the base representation). On the spin j of a representation (corresponding to a horizontal weight $\lambda_1 = j\alpha_1$, these Weyl reflections act as follows:

$$s^\rho_{\alpha_1+l\delta}(j) = -l(k_{su(2)}+2) - j - 1$$
$$s^\rho_{-\alpha_1+l\delta}(j) = l(k_{su(2)}+2) - j - 1. \tag{A.2}$$

Following [34] we also define an operation S on the spin and the level of the affine algebra which acts as $(j, k_{su(2)}) \mapsto (-j-1, -k_{su(2)}-4)$.

A.1 Flipping

The statements on the structure of the embedding of Verma modules derived in [34], and restricted to the case of relevance here, can now be made more transparent, using the above notations.

First of all, the main theorem on the embedding structure in [34], in paragraph 2, case A, pertains to the case where the level of the sl_2 algebra, which we have named $k_{su(2)}$ is larger than -2. However, the case of interest to us in the bulk can be reduced to this case via the operation S. Indeed, consider a spin and bosonic $su(2)$ level $(j_n, -k-2)$ as in the bulk of our paper. (Note that $k_{su(2)} = -k-2$, where k is the supersymmetric $\mathrm{SL}(2,R)$ level used in the bulk of the paper.) By acting with the operation S, we find the spin and level $(-j_n-1, k-2)$. Since our supersymmetric level k is larger than zero, we do find that the new bosonic level $k-2$ is larger than -2, and that we now therefore fall into the framework of the embedding diagrams discussed in paragraph 2, case A of [34].

A.2 Constructing the embedding diagram

The results of Malikov [34] then permit us to construct the embedding diagrams for all cases. In particular, we concentrate on the most difficult case of $k = p/q$ fractional (with p, q both strictly positive mutually prime integers). First note that the Kaz-Kazhdan equation can be written more invariantly as being satisfied for an affine weight λ when:

$$2(\lambda+\rho,\alpha) = m(\alpha,\alpha) \tag{A.3}$$

for some positive root α and $m \in N_0$ (i.e. m a strictly positive integer). This translates directly into the equation used in the bulk of the paper. However, following [34], we will now consider solutions to this equation for m any integer. In other words, we will study solutions to the equation:

$$-2j_n - 1 = -m - kn \tag{A.4}$$

which is the Kaz-Kazhdan equation for the flipped spin defined above, and with respect to the flipped supersymmetric level k, and this for any integers m and n. Pick now the smallest positive integer n such that the equation is satisfied, and the smallest strictly negative integer n such that the equation holds. For the values j_n defined in the bulk of the paper, in equation (3.30), it is not hard to see that these values correspond to $s_n, s_n - q$ and $s_n + q, s_n$ for j_n positive and negative respectively. These, via the Kac-Kazhdan equations above correspond to positive affine roots which are, for $2j_n + 1$ positive:

$$\begin{aligned} \alpha_1 + s_n\delta \\ -\alpha_1 + (q - s_n)\delta \end{aligned} \tag{A.5}$$

while for $2j_n + 1$ negative, we consider the positive roots:

$$\begin{aligned} \alpha_1 + (q + s_n)\delta \\ -\alpha_1 - s_n\delta. \end{aligned} \tag{A.6}$$

The results of [34] show that the embedding diagrams of the Verma module with spin j can be constructed from an initial Verma module j_0 by ρ-centered Weyl-reflecting from this initial point, and adding a node to the embedding diagram for each step on the way to the destination j. The node j_0 corresponds to the unique spin satisfying the equations:

$$0 \le -2j_0 - 1 + s_n k \le p \tag{A.7}$$

respectively

$$0 \le 2j_0 + 1 - s_n k \le p, \tag{A.8}$$

and lying in the orbit of j under the above ρ-centered Weyl reflections.

Since this prescription is rather abstract (although already more concrete than the original one [34]), let us first illustrate this in the case where the level k is integer, and see how we recuperate the diagrams used in the bulk of the paper.

The case of k integer

For k integer, we have that $q = 1$, and that for $2j_n + 1$ positive, say, we have that $s_n = 0$. Consequently, the allowed ρ-centered Weyl reflections are those with respect to α_1 and $\alpha_1 + \delta$. These are the standard ρ-centered Weyl reflections, and those are precisely the ones we used to construct the diagrams in the case of integer level. Moreover, the condition on the final spin becomes $0 \le -2j_0 - 1 \le k = p$ which is precisely what we found in the bulk of the paper.

The general case

In the general case too, one can show with a bit more tedious but straightforward calculations that the above systematic construction of the embedding diagrams agrees on the nose with the analysis in terms of singular vectors only performed in the bulk of the paper. In fact, this justifies the approach in the bulk of the paper, via the detailed proof of Malikov of the embedding structure of all Verma modules for the affine sl_2 algebra.

B. A case study

As we have seen, for j-values that are not of the form $2j+1 \in k\mathbb{Z}$, the sub-module structure is not linear. The simplest example of a non-linear embedding of sub-modules occurs when the embedding diagram has only four entries (the "rhombus" diagram). From the general embedding diagram (3.9) for integer k, we see that this happens for $j = j_0 \pm k$. Let us discuss each of these cases in turn and demonstrate explicitly how one would compute cohomology inductively.

B.1 The case $j = j_0 + k$

For $j > -\frac{1}{2}$, the Wakimoto modules are isomorphic to the Verma modules and we can read off the submodule structure from [34] to be (with $W_j = V_j$)

$$\begin{tikzcd} & V_{-j_0-1} \arrow[dr] & \\ V_{j_0+k} \arrow[ur] \arrow[dr] & & V_{j_0} \\ & V_{-j_0-k-1} \arrow[ur] & \end{tikzcd} \tag{B.1}$$

In order to find the cohomology of Q in this module W_j, we have to first find the cohomology of the modules W_{-j_0-1} and W_{-j_0-k-1}. However, these have linear sub-module structure and their cohomology can be computed using the techniques of Frenkel in [15]. Let us illustrate this in the case of $W_{-j_0-1} = V_{-j_0-1}$, which has the embedding structure

$$V_{-j_0-1} \longrightarrow V_{j_0}\,. \tag{B.2}$$

Now, it follows that the following sequence of modules is a short exact sequence:

$$0 \longrightarrow I_{-j_0-1} \longrightarrow W^*_{-j_0-1} \longrightarrow W_{j_0} \longrightarrow 0\,. \tag{B.3}$$

So far we have only focused on the modules over the SL$(2,\mathbb{R})$ current algebra. In order to compute the cohomology of the Q operator in (2.16), we have to tensor the SL$(2,\mathbb{R})$ modules with the Fock modules over the (b,c) and X algebra. Although we will suppress these tensor products while writing out the sequences of modules, it is good to keep in mind that the states (observables) we will find as cohomology elements belong to the Hilbert space in (2.12).

The corresponding long exact sequence of cohomology is given by

$$\ldots \longrightarrow H^q(I_{-j_0-1}) \longrightarrow H^q(W^*_{-j_0-1}) \longrightarrow H^q(W_{j_0}) \longrightarrow H^{q+1}(I_{-j_0-1}) \longrightarrow \ldots \tag{B.4}$$

Using (3.1) and (3.6), our long exact sequence collapses to the following short exact sequences

$$\begin{aligned} 0 \longrightarrow H^1(I_{-j_0-1}) \longrightarrow \{-j_0-1\} \longrightarrow 0 \\ 0 \longrightarrow \{j_0\} \longrightarrow H^2(I_{-j_0-1}) \longrightarrow 0 \end{aligned} \tag{B.5}$$

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all other cohomology groups being trivial. We thus get $H^1(I_{-j_0-1}) = \{-j_0 - 1\}$ and $H^2(I_{-j_0-1}) = \{j_0\}$. Now consider the short exact sequence dual to the one considered above

$$0 \longrightarrow W^*_{j_0} \longrightarrow W_{-j_0-1} \longrightarrow I_{-j_0-1} \longrightarrow 0\,. \tag{B.6}$$

Once again, considering the corresponding long exact sequence and using the results derived so far, we get the collapsed short exact sequences for the cohomology

$$\begin{aligned} 0 \longrightarrow \{j_0\} \longrightarrow H^1(W_{-j_0-1}) \longrightarrow \{-j_0-1\} \longrightarrow 0 \\ 0 \longrightarrow H^1(W_{-j_0-1}) \longrightarrow \{j_0\} \longrightarrow 0 \end{aligned} \tag{B.7}$$

Similar manipulations can be done for $j = -k - j_0 - 1$ and we get the following result for the cohomology:

$$\begin{aligned} \text{For } j = -j_0 - 1, \quad & H^1(W_{-j_0-1}) = \{-j_0 - 1, j_0\} \\ & H^2(W_{-j_0-1}) = \{j_0\} \\ \text{For } j = -k - j_0 - 1, \quad & H^1(W_{-k-j_0-1}) = \{-k - j_0 - 1, j_0\} \\ & H^0(W_{-k-j_0-1}) = \{j_0\} \end{aligned} \tag{B.8}$$

We now have the ingredients necessary to compute the cohomology of W_{j_0+k} whose sub-module structure is shown in (B.1). We claim that the following sequences are exact

$$0 \longrightarrow I_{j_0} \longrightarrow V_{-k-j_0-1} \oplus V_{-j_0-1} \longrightarrow \mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1}) \longrightarrow 0 \tag{B.9}$$

$$0 \longrightarrow \mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1}) \longrightarrow V_{j_0+k} \longrightarrow I_{j_0+k} \longrightarrow 0 \tag{B.10}$$

In all, there are four unknown cohomologies, those of the modules V_{j_0+k}, I_{j_0+k}, and the cohomology of the image module that appears in sequence (B.9) and its dual. Using these two short exact sequences and their dual sequences, it is possible to compute the cohomology of all four modules.

We first solve for the cohomology of $\mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1})$. Using the fact that

$$V_{-j_0-k-1} = W^*_{-j_0-k-1} \quad \text{and} \quad V_{-j_0-1} = W_{-j_0-1}\,, \tag{B.11}$$

the long exact sequence corresponding to the short exact sequence in (B.9) collapses to

$$\begin{aligned} 0 \longrightarrow H^0(\mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1})) \longrightarrow \{j_0\} \longrightarrow \{-k - j_0 - 1, -j_0 - 1, j_0\} \longrightarrow \\ \longrightarrow H^1(\mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1})) \longrightarrow 0 \end{aligned} \tag{B.12}$$

$$0 \longrightarrow \{j_0\} \longrightarrow H^2(\mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1})) \longrightarrow 0 \tag{B.13}$$

This leads to

$$\begin{aligned} H^1(\mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1})) = \{-k - j_0 - 1, -j_0 - 1\} \\ H^2(\mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1})) = \{j_0\}\,. \end{aligned} \tag{B.14}$$

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Let us now consider the sequence dual to (B.9). Using similar techniques, we find that the dual of the image module has the same cohomology, with H^0 replaced by H^2. Let us now apply these results to the dual of the exact sequence in (B.10):

$$0 \longrightarrow I_{j_0+k} \longrightarrow V^*_{j_0+k} \longrightarrow \mathrm{Im}^*(V_{-k-j_0-1} \oplus V_{-j_0-1}) \longrightarrow 0\,. \tag{B.15}$$

Since, $V_{j=j_0+k} = W_{j_0+k}$, the corresponding long exact sequence collapses to

$$\begin{aligned} 0 \longrightarrow \{j_0\} \longrightarrow H^1(I_{j_0+k}) \longrightarrow \{j_0+k\} \longrightarrow 0 \\ 0 \longrightarrow \{-k-j_0-1, -j_0-1\} \longrightarrow H^2(I_{j_0+k}) \longrightarrow 0 \end{aligned} \tag{B.16}$$

from which we get $H^1(I_{j_0+k}) = \{j_0, j_0+k\}$ and $H^2(I_{j_0+k}) = \{-k-j_0-1, -j_0-1\}$. Now, writing out the long exact sequence corresponding to the short exact sequence in (B.10) and using all the results obtained so far, we get the following non-trivial long exact sequence

$$\begin{aligned} 0 \longrightarrow \{-k-j_0-1, -j_0-1\} \longrightarrow H^1(W_{j_0+k}) \longrightarrow \{j_0, j_0+k\} \longrightarrow \{j_0\} \longrightarrow \\ \longrightarrow H^2(W_{j_0+k}) \longrightarrow \{-k-j_0-1, -j_0-1\} \longrightarrow 0 \end{aligned} \tag{B.17}$$

The long sequence breaks as shown in the equation into two subsequences, which can be solved to obtain

$$\begin{aligned} H^1(W_{j_0+k}) &= \{-k-j_0-1, -j_0-1, j_0+k\} \\ H^2(W_{j_0+k}) &= \{-k-j_0-1, -j_0-1\} \end{aligned} \tag{B.18}$$

B.2 The case $j = j_0 - k$

As mentioned at the beginning of this section, there is another value of j for which the sub-module structure is the same as in (B.1), $j = j_0 - k < 0$. However, since for negative j, Wakimoto modules are isomorphic to dual Verma modules [15], the directions in the arrows of the embedding diagram for the Wakimoto module are opposite to what was considered earlier (since $W_{j_0-k} = V^*_{j_0-k}$):

$$\begin{tikzcd} & V^*_{-j_0-1} \arrow[dl] & \\ V^*_{j_0-k} & & V^*_{j_0} \arrow[ul] \arrow[dl] \\ & V^*_{-k-j_0-1} \arrow[ul] & \end{tikzcd} \tag{B.19}$$

Since this case is different from the $j > 0$ case, let us go through the exercise of computing cohomology explicitly. The short exact sequences in (B.9) and (B.10) remain the same, with $j_0 + k$ replaced by $j_0 - k$. However, when writing out the long exact sequence of cohomologies, it is important to use the equality $V_{j_0-k} = W^*_{j_0-k}$. With this in mind, let us proceed as before.

The computation of the cohomology of $\mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1})$ using the sequence in (B.10) remains unchanged. We are thus left with the sequence

$$0 \longrightarrow \mathrm{Im}(V_{-k-j_0-1} \oplus V_{-j_0-1}) \longrightarrow W^*_{j_0-k} \longrightarrow I_{j_0-k} \longrightarrow 0\,. \tag{B.20}$$

The corresponding long exact sequence breaks up into

$$\begin{aligned} 0 \longrightarrow H^0(I_{j_0-k}) \longrightarrow \{-k-j_0-1, -j_0-1\} \longrightarrow 0 \\ 0 \longrightarrow \{j_0-k\} \longrightarrow H^1(I_{j_0-k}) \longrightarrow \{j_0, -j_0-1\} \longrightarrow 0 \end{aligned} \tag{B.21}$$

leading to $H^0(I_{j_0-k}) = \{-k-j_0-1, -j_0-1\}$ and $H^1(I_{j_0-k}) = \{j_0-k, j_0\}$. Consider now the sequence dual to (B.20)

$$0 \longrightarrow I_{j_0-k} \longrightarrow W_{j_0-k} \longrightarrow \mathrm{Im}^*(V_{-k-j_0-1} \oplus V_{-j_0-1}) \longrightarrow 0\,. \tag{B.22}$$

The long exact sequence is of the form

$$\begin{aligned} 0 \longrightarrow \{-k-j_0-1, -j_0-1\} \longrightarrow H^0(W_{j_0-k}) \longrightarrow \{j_0\} \longrightarrow \\ \longrightarrow \{j_0, j_0-k\} \longrightarrow H^1(W_{j_0-k}) \longrightarrow \{-k-j_0-1, -j_0-1\} \longrightarrow 0 \end{aligned} \tag{B.23}$$

which breaks into two shorter sequences (as indicated in the diagram). Solving for the cohomology groups, we get

$$\begin{aligned} H^1(W_{j_0-k}) = \{j_0-k, -k-j_0-1, -j_0-1\} \\ H^0(W_{j_0-k}) = \{-k-j_0-1, -j_0-1\} \end{aligned} \tag{B.24}$$

The notable feature of this analysis is the following:

- The spin j-values of the cohomology elements correspond to the singular vectors that appear in the original module. For both $j = j_0 \pm k$, the module generated by $j = j_0$ is not a submodule (i.e. j_0 not a singular vector within the original module). This can be explicitly checked by finding solutions to the Kac-Kazhdan equation in (3.2).

This feature generalizes to the generic case with more nodes and also to the rational case, and has inspired our ansatz in (3.44) which we have proven in the main text using the method of induction.

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